Bank A →

Bank B →

Chapter 4
Section 4.1
Warm-up Problem A. Suppose you have 100 dollars to deposit into a savings account. If you put your money into Bank A, they will deposit an additional 10 dollars per year into your account. If you put your money into Bank B, they will increase your balance by 10 percent per year.

(a) How much money would you have after one year if you put your money into Bank A? How about Bank B?

$100 + 10 = 110$ (original) $\qquad 100 + (.1)(100) = 110$ (originally)

(b) After two years?

$100 + 10 + 10 = 120 \qquad 110 + (.1)(110) = 121$

(c) After three years?

$100 + 10 + 10 + 10 = 130 \qquad 121 + (.1)121 = 133.1$

(d) Which bank should you use?

$\boxed{\text{Bank B}}$

Main Topic # 1: [Exponential Functions]

The type of functions we will be looking at today are called Exponential Functions, these are functions of the form:

Exponential Functions (The 'Look' Definition)

Exponential functions are functions with the following form

$$f(x) = b(m^x)$$

where $m > 0$

Notice we can find the y-intercept by setting $x = 0$, which fill give us $(0, f(0)) = (0, b)$ as $m^0 = 1$.

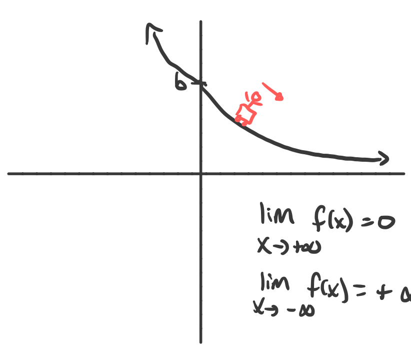
Also, notice there is **NO** x-intercept since with the condition $m > 0$ then for no value of x does m^x become zero or negative at that matter!

Also, nothing goes wrong when we raise to any number so the **Domain** is all real numbers i.e. $(-\infty, \infty)$.

Sketching Exponentials $b > 0$

Exponential functions look like:

$(b > 0)$
$0 < m < 1$



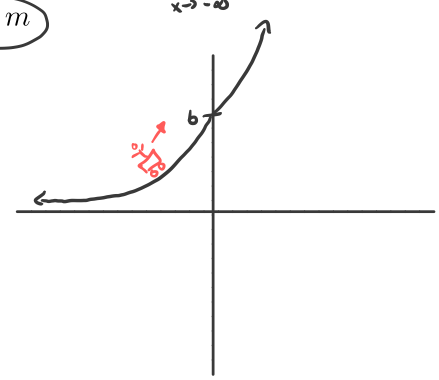
$\lim_{x \to +\infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = +\infty$

$(b > 0)$
$1 < m$

$\lim_{x \to +\infty} f(x) = +\infty$

$\lim_{x \to -\infty} f(x) = 0$



The above pictures show us that when $0 < m < 1$ then the function is **decreasing** and when $1 < m$ then the function is **increasing**.

Also we see from these pictures that exponential functions have a **horizontal asymptote** as $y = 0$ (i.e. the x-axis) but NEVER crosses it!

This also shows us that the **Range** of this function is all positive numbers that is $(0, \infty)$.

Main Topic # 2: [Exponentials and Change]

Recall that we saw a line was the only type of function which had the same average rate of change no matter what two points we plugged in. An *exponential function* is the only function that has constant Percent Change no matter where you start. **Unlike** average rate of change percent change doesn't take in two "random points" it only uses two points whose x-value **differs by 1-unit!**

Percent Change

The percent change of a function $f(x)$ at $x = a$ is the following formula:

$$\frac{f(a+1) - f(a)}{f(a)} = \frac{\text{new} - \text{old}}{\text{old}}$$

Percent Change Example

Find the percent change for the following function $f(x)$ at $x = 3$, given the following data:

x	3	4
$f(x)$	12	14

Work done here:

$$\frac{f(4) - f(3)}{f(3)} = \frac{14-12}{12} = \frac{2}{12} = \frac{1}{6}$$

Exponential Function (The Change Definition)

We call the constant percent change of an exponential function, $f(x)$, the rate and usually denote it as $r = \frac{f(a+1)-f(a)}{f(a)}$, and in the usual manner if we denote b as the y-intercept (just like we do for lines) then we get the following formula for any exponential function:

$$f(x) = b(\overbrace{1+r}^{m})^x$$

Notice, the comparison of the two forms specifically that $1 + r = m$. Therefore, when $-1 < r < 0$ then $f(x)$ is **decreasing** and when $r > 0$ then $f(x)$ is **increasing**.

$$\frac{f(a+1) - f(a)}{f(a)} < 0$$

When an exponential is increasing we say the function is growing (growth).

When an exponential function is decreasing we say the function is decaying (decay)

Comparing Rates

Just Like in lines the rate (i.e. the constant percent change) tells us how 'steep' the function is (i.e. how rapidly the function is increasing or decreasing).

Let $f(x) = b_1(1+r_1)^x$ and $g(x) = b_2(1+r_2)^x$ then when $0 < r_2 < r_1$ we have that:

$f(x)$ grows faster than $g(x)$

Similarly, when $r_1 < r_2 < 0$ we have that

$f(x)$ decays faster than $g(x)$

Learning Outcome # 1: [Be able to identify when a function has constant percent change]

Problem 1. The two tables below give some values for a function $f(t)$ and $g(t)$. Could $f(t)$ and/or $g(t)$ be exponential? Why or why not?

t	1	2	3	4	5
$f(t)$	14	18	24	31	40

$$\frac{f(2)-f(1)}{f(1)} = \frac{18-14}{14} = \frac{4}{14} = \frac{2}{7}$$

$\neq$ not equal so not exponential

$$\frac{f(3)-f(2)}{f(2)} = \frac{24-18}{18} = \frac{6}{18} = \frac{3}{9} = \frac{1}{3}$$

t	1	2	3	4	5
$g(t)$	15	18	21.6	25.82	30.984

exponential

$$\frac{f(2)-f(1)}{f(1)} = \frac{18-15}{15} = \frac{3}{15} = \frac{1}{5} = 0.2$$

$$\frac{f(4)-f(3)}{f(3)} = \frac{25.82-21.6}{21.6} = 0.2$$

$$\frac{f(3)-f(2)}{f(2)} = \frac{21.6-18}{18} = 0.2$$

$$\frac{f(5)-f(4)}{f(4)} = \frac{30.984-25.82}{25.82} = 0.2$$

Learning Outcome # 2: [Given the rate and y-intercept write the function for an exponential]

Problem 2. Mildred collects hats. In 2014, she began with 25 hats, and she plans to increase her collection by 25% per year. Write a formula for $h(t)$, the number of hats Mildred will have t years after 2014. How large will her collection be in 2020?

$t=6$

$r = +0.25$ 25%

Since increasing

$h(0) = 25$

$h(t) = 25(1+0.25)^t$

$= 25(1.25)^t$

$h(6) = 25(1.25)^6$

~~$\not{A}$~~ → fix This in meeting

Learning Outcome # 3: [Writing an exponential function given two points]
Problem 3. Consider the points $(0, 3)$ and $(1, 7)$ on the function $f(x)$. Given that the function is an exponential write the formula for $f(x)$.

$$r = \frac{f(1) - f(0)}{f(0)} = \frac{7-3}{3} = \frac{4}{3}$$

y-int → $b = 3$

$$f(x) = 3\left(1 + \frac{4}{3}\right)^x$$

Learning Outcome # 4: [Identifying growth and decay]
Problem 4. The grades of six students are given by the equations below, where time, t, is measured in weeks after the first midterm exam.

(i) $P = 97(1.001)^t$	(ii) $P = 58(1.05)^t$	(iii) $P = 72(0.9)^t$
(iv) $P = 85(0.99)^t$	(v) $P = 79(1.03)^t$	(vi) $P = 85(0.89)^t$

(a) Alex's grade has been decreasing since the first exam; Which formula(s) could represent her grade and why?

(iii) (iv) (vi) All have base less than 1

(b) Which student's grade is falling the fastest?

(vi) most negative "r" i.e. smallest base!

(c) Umar's grade has been rising since the first exam; Which formula(s) could represent his grade and why?

(i) (ii) (v) since base larger than 1

(d) Which student's grade is rising the fastest?

(ii) has largest base i.e. largest rate

(e) Karan came out of the first exam with a 85% in the class; which formula(s) could represent his grade?

(iv) & (vi) Since y-int is 85 (i.e. initial value) investments mean increase

Learning Outcome # 5: [Word Problems]

Problem 5. Suppose an investment of \$800 earns 7% interest per year. Come up with a formula for how much money you will have in t years.

$$M(t) = 800(1+0.07)^t$$

Problem 6. Sally makes a deal with her mother. If Sally does all her chores today, she will get half a cookie. If she does her chores on the second day as well, she will get one cookie. If she does her chores on the third day, she will get two cookies. Each day in a row that she does her chores, Sally will get double the amount of cookies as the previous day. Let $C(n)$ give the number of cookies Sally gets on day n. What is the growth rate? What is the initial value? Write a formula for $C(n)$. How many cookies does Sally get on day 30? Did Sally make a good deal?

$$C(n) = \tfrac{1}{2}(1+1)^n = \tfrac{1}{2}(2)^n$$

$$\frac{\overset{\text{new}}{2} - \overset{\text{old}}{1}}{\underset{\text{old}}{1}} = 1$$

Problem 7. Your boss proposes two different plans for giving raises. Suppose you currently earn \$35,000 per year. In the first plan, you earn an additional \$1,000 each year. In the second plan, you increase your salary by 3% each year. Which plan do you think is better? Check your guess by computing your salary after 5 years and after 10 years under both plans.

$$P_1 = 1{,}000 \cdot x + 35{,}000 \qquad P_2 = 35{,}000(1+.03)^x$$

Problem 8. Suppose a city has a population of 400,000 at time $t = 0$, where t is the number of years after 1990.

look exponentials grow faster than lines

(a) If the city's population decreases by 2,500 people per year, find a formula for $P(t)$, the city's population t years after 1990.

$$-2500x + 400000$$

(b) If the city's population decreases by 6% per year, find a formula for $P(t)$, the city's population t years after 1990.

$$400000(1-0.06)^t$$

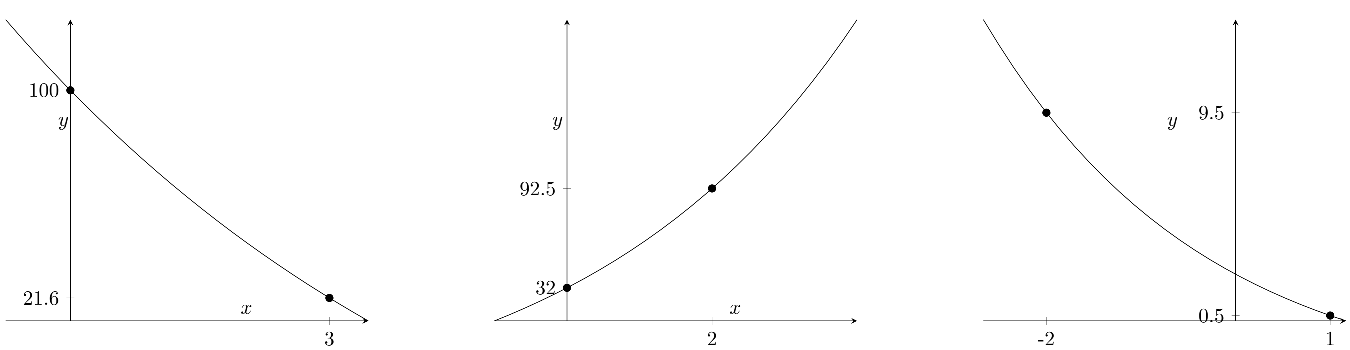
Bonus Question! free 5 points

Also very important to calculus!

Note these are Not 1 apart! Can't use percent change

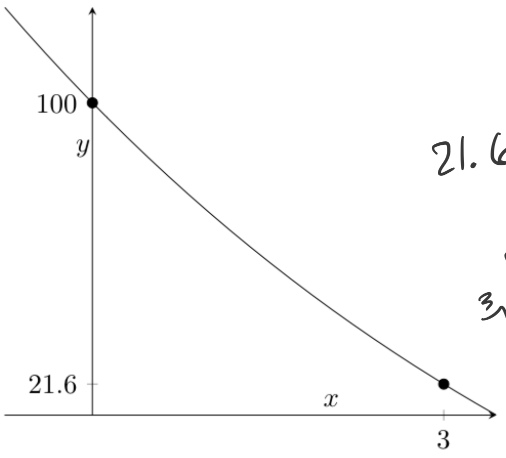
Learning Outcome # 6: [Identify Exponential functions from graphs]

Problem 9. Write an equation for each of the exponential functions whose graphs are shown below.



hey look!

Solution on next page!



$$f(x) = 100m^x$$

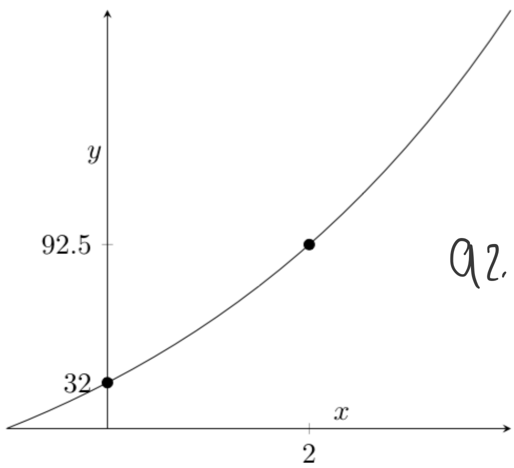
$$21.6 = f(3) = 100m^3$$

$$\sqrt[3]{\frac{21.6}{100}} = \sqrt[3]{m^3}$$

$$m = \sqrt[3]{0.216}$$

$$f(x) = 100\left(\sqrt[3]{0.216}\right)^x$$

$$= 100\,(0.216)^{\frac{x}{3}}$$



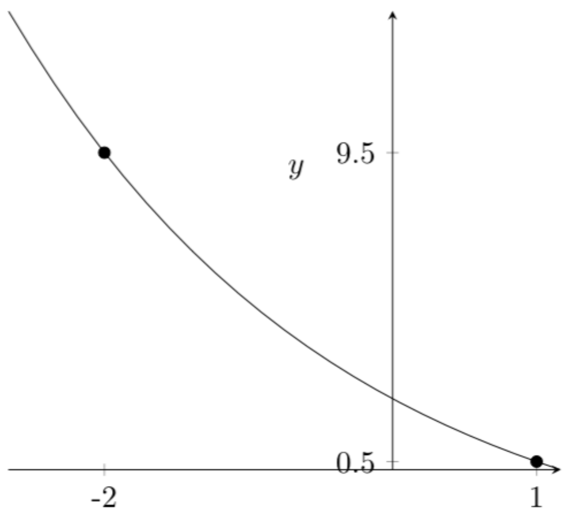
$$f(x) = 32m^x$$

$$92.5 = f(2) = 32m^2$$

$$\sqrt{\frac{92.5}{32}} = \sqrt{m^2}$$

$$m = \sqrt{\frac{92.5}{32}}$$

$$f(x) = 32\left(\frac{92.5}{3}\right)^{\frac{x}{2}}$$



$$f(x) = (0.03)\,19^x$$

$$f(x) = bm^x$$

$$f(1) = \frac{1}{2} = bm$$

$$f(-2) = 9.5 = bm^{-2}$$

$$\frac{\left(\frac{1}{2}\right)}{9.5} = \frac{bm}{bm^{-2}}$$

$$f(1) = \frac{1}{2} = b(19)$$

$$\Rightarrow \frac{1}{2(9.5)} = m^{-1}$$

$$0.03 = \frac{1}{2(19)} = b$$

$$\Rightarrow m = 2(9.5) = 19$$

Main Topic # 3: [Euler's Number] There is a special number discovered by the mathematician named Euler (pronounced 'Oil-er') we denote the number by the lowercase letter: e

e is Irrational

It is an example of an irrational number, recall this means it is a decimal that never repeats or ends...

Here are some of the digits:

e =2.71828182845904523536028747135266249775724709369995957496696762772407663035 3...

Euler discovered this number because using some technical techniques you will learn in Calculus but we love the number e because it has some awesome properties. Specifically when we consider the function $f(x) = e^x$ then average rate of change has some peculiar behavior.

☆ know this # i.e. if I write e^x ↗ know it's some number (2.172) raised to the x ☆

The next exercise is designed to show you this:

don't worry about this problem unless you are really Bored

Problem 10. Let $f(x) = e^x$.

(a) Compute the average rate of change of $f(x)$ on the intervals $[0, 2], [0.75, 1.25]$, and $[0.99, 1.01]$. Compare these values to e^1. What do you notice?

(b) Compute the average rate of change of $f(x)$ on the intervals $[1, 3], [1.75, 2.25]$, and $[1.99, 2.01]$. Compare these values to e^2. What do you notice?

(c) Compute the average rate of change of $f(x)$ on the intervals $[6, 8], [6.75, 7.25]$, and $[6.99, 7.01]$. Compare these values to e^7. What do you notice?

(d) Make a prediction about the average rate of change of $f(x)$ at any particular x value.